

# Reliability study of series and parallel systems of heterogeneous component lifetimes under proportional odds model

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## Abstract

In this paper, we investigate various stochastic orderings for series and parallel systems with independent and heterogeneous components having lifetimes following the proportional odds model. We also investigate comparisons between system with heterogeneous components and that with homogeneous components. This paper also studies relative ageing orders for two systems in the framework of components having lifetimes following the proportional odds model.

Keywords: Majorization, Schur-concave function, Schur-convex function, Stochastic order, Relative ageing.

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## 1 Introduction

There is an extensive literature on different stochastic orderings among order statistics where the observations come from different family of distributions. Some of these contributions are due to Balakrishnan and Zhao (2013), Bon and Păltănea (2006), Dykstra et al. (1997), Fang and Zhang (2012, 2015), Gupta et al. (2015), Khaledi and Kochar (2000), Khaledi et al. (2011), Kochar and Xu (2007a,b), Kundu et al. (2016), Li and Li (2016), Misra and Misra (2013), Pledger and Proschan (1971), Zhao and Balakrishnan (2011, 2012). A one-to-one correspondence between an order statistic and the lifetime of a  $k$ -out-of- $n$  system is well known. A  $k$ -out-of- $n : G$  system (generally called  $k$ -out-of- $n$  system) is a system consisting of  $n$  components which survives as long as at least  $k$  of the  $n$  components survive. Let  $X_{k:n}$  be the  $k$ th

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smallest order statistic corresponding to the random variables  $X_1, X_2, \dots, X_n$ ,  $k = 1, 2, \dots, n$ . Then the lifetime of a  $(n - k + 1)$ -out-of- $n$  :  $G$  system corresponds to the order statistic  $X_{k:n}$ . So,  $X_{n-k+1:n}$  represents lifetime of an  $k$ -out-of- $n$  :  $G$  system. In particular,  $X_{1:n}$  and  $X_{n:n}$  represent lifetimes of the series and the parallel systems, respectively.

The proportional odds (PO) model introduced by Bennet (1983) is a very important model in survival analysis context, mainly for its property of convergent hazard functions. The PO model, as discussed by Bennet (1983) and latter by Kirmani and Gupta (2001) guarantees that the ratio of hazard rates converges to unity as time tends to infinity. This is in contrast to the proportional hazards model where the ratio of the hazard rates remains constant with time. The convergent property of hazard functions makes the PO model reasonable in many practical applications as discussed by Bennet (1983), Kirmani and Gupta (2001) and Rossini and Tsiatis (1996). They also noticed that assumption of constant hazard ratio is unreasonable in many practical cases. For more applications of PO model one may refer to Collett (2004), Dinse and Lagakos (1983), Kirmani and Gupta (2001), Pettitt (1984).

Let  $X$  and  $Y$  be two random variables with distribution functions  $F(\cdot)$ ,  $G(\cdot)$ , survival functions  $\bar{F}(\cdot)$ ,  $\bar{G}(\cdot)$ , probability density functions  $f(\cdot)$ ,  $g(\cdot)$  and hazard rate functions  $r_X(\cdot) = f(\cdot)/\bar{F}(\cdot)$ ,  $r_Y(\cdot) = g(\cdot)/\bar{G}(\cdot)$  respectively. Let the odds functions of  $X$  and  $Y$  be defined respectively by  $\theta_X(t) = \bar{F}(t)/F(t)$  and  $\theta_Y(t) = \bar{G}(t)/G(t)$ . The random variables  $X$  and  $Y$  are said to satisfy PO model with proportionality constant  $\alpha$  if  $\theta_Y(t) = \alpha\theta_X(t)$ . It is observed that, in terms of survival functions, the PO model can be represented as

$$\bar{G}(t) = \frac{\alpha\bar{F}(t)}{1 - \bar{\alpha}\bar{F}(t)}, \quad (1.1)$$

where  $\bar{\alpha} = 1 - \alpha$ . From the above representation we have

$$\frac{r_Y(t)}{r_X(t)} = \frac{1}{1 - \bar{\alpha}\bar{F}(t)} = \frac{G(t)}{F(t)},$$

so that the hazard ratio is increasing (resp. decreasing) for  $\alpha > 1$  (resp.  $\alpha < 1$ ) and it converges to unity as  $t$  tends to  $\infty$ . Also the model (1.1), with  $0 < \alpha < \infty$ , gives a method of introducing new parameter  $\alpha$  to a family of distributions for obtaining more flexible new family of distributions as discussed by Marshall and Olkin (1997). The family of distributions so obtained is also known as Marshall-Olkin family of distributions or Marshall-Olkin extended distributions (for details see Marshall and Olkin (1997, 2007) and Cordeiro et al. (2014) among others).

Stochastic comparison of different systems with components following proportional hazard rates (PHR) model have been discussed by Pledger and Proschan (1971), Dykstra et al. (1997), Khaledi and Kochar (2000), Kochar and Xu (2007a,b), Li and Li (2016) among others. How-

ever, not much work have been done on stochastic comparison of systems with components following PO model. In this paper, we investigate stochastic comparisons of series and parallel systems with heterogeneous components having lifetimes following the PO model. We also obtain some stochastic comparison results between system with heterogeneous components and that with homogeneous ones. The comparisons are done with respect to the usual stochastic ordering, the hazard rate ordering, the reversed hazard rate ordering, the likelihood ratio ordering, and the relative ageing orderings.

Throughout the paper, by  $a =^{sign} b$  we mean that  $a$  and  $b$  have the same sign and by  $a =^{def} b$  we mean that  $b$  is defined as  $a$ . We also write  $\mathbb{R} = (-\infty, \infty)$  and  $\mathbb{R}^+ = \{x : x > 0\}$ .

## 2 Definitions and Preliminaries

Majorization is a preorder on vectors of real numbers. Let  $I \subseteq \mathbb{R}$  denote a subset of the real line. Further let, for any vector  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ ,  $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$  denote the increasing arrangement of the components of the vector  $\mathbf{x}$ . Below we give a couple of definitions to be used throughout the paper.

**Definition 2.1** Let  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in I^n$  and  $\mathbf{y} = (y_1, y_2, \dots, y_n) \in I^n$  be any two vectors.

(i) The vector  $\mathbf{x}$  is said to majorize the vector  $\mathbf{y}$  (written as  $\mathbf{x} \succeq^m \mathbf{y}$ ) if (cf. Marshall et al., 2011)

$$\sum_{i=1}^j x_{(i)} \leq \sum_{i=1}^j y_{(i)}, \text{ for all } j = 1, 2, \dots, n-1, \text{ and } \sum_{i=1}^n x_{(i)} = \sum_{i=1}^n y_{(i)}.$$

(ii) The vector  $\mathbf{x}$  is said to weakly supermajorize the vector  $\mathbf{y}$  (written as  $\mathbf{x} \succeq^w \mathbf{y}$ ) if (cf. Marshall et al., 2011)

$$\sum_{i=1}^j x_{(i)} \leq \sum_{i=1}^j y_{(i)}, \text{ for all } j = 1, 2, \dots, n.$$

(iii) The vector  $\mathbf{x}$  is said to weakly submajorize the vector  $\mathbf{y}$  (written as  $\mathbf{x} \succeq_w \mathbf{y}$ ) if (cf. Marshall et al., 2011)

$$\sum_{i=j}^n x_{(i)} \geq \sum_{i=j}^n y_{(i)}, \text{ for all } j = 1, 2, \dots, n.$$

(iv) The vector  $\mathbf{x}$  is said to be  $p$ -larger than the vector  $\mathbf{y}$  (written as  $\mathbf{x} \succeq^p \mathbf{y}$ ) if (cf. Bon and Păltănea,

1999)

$$\prod_{i=1}^j x_{(i)} \leq \prod_{i=1}^j y_{(i)}, \text{ for all } j = 1, 2, \dots, n.$$

(v) The vector  $\mathbf{x}$  is said to reciprocally majorize the vector  $\mathbf{y}$  (written as  $\mathbf{x} \stackrel{rm}{\succeq} \mathbf{y}$ ) if (cf. Zhao and Balakrishnan, 2009)

$$\sum_{i=1}^j \frac{1}{x_{(i)}} \geq \sum_{i=1}^j \frac{1}{y_{(i)}}, \text{ for all } j = 1, 2, \dots, n.$$

It can be seen that

$$\mathbf{x} \stackrel{m}{\succeq} \mathbf{y} \Rightarrow \mathbf{x} \stackrel{w}{\succeq} \mathbf{y} \Rightarrow \mathbf{x} \stackrel{p}{\succeq} \mathbf{y} \Rightarrow \mathbf{x} \stackrel{rm}{\succeq} \mathbf{y}.$$

**Remark 2.1** Definition 3.1(i) can equivalently be written as

$$x \stackrel{m}{\succeq} y \text{ if } \sum_{i=1}^j x_{[i]} \geq \sum_{i=1}^j y_{[i]}, \text{ for all } j = 1, 2, \dots, n-1, \text{ and } \sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]},$$

where  $x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[n]}$  is a decreasing arrangement of  $x_1, x_2, \dots, x_n$ .

**Definition 2.2** A function  $\phi : I^n \rightarrow \mathbb{R}$  is said to be Schur-convex (resp. Schur-concave) on  $I^n$  if

$$\mathbf{x} \stackrel{m}{\succeq} \mathbf{y} \Rightarrow \phi(\mathbf{x}) \geq (\text{resp. } \leq) \phi(\mathbf{y}).$$

Below we give some definitions of stochastic orders. The details of usual stochastic order, failure rate order, reversed hazard rate order and likelihood ratio orders may be obtained in Shaked and Shanthikumar (2007), whereas the relative ageing ordering with respect to hazard rate is given in Sengupta and Deshpande (1994), and Rezaei et al. (2015) discuss the relative ageing ordering with respect to reversed hazard rate.

**Definition 2.3** Let  $X$  and  $Y$  be two absolutely continuous random variables with cumulative distribution functions  $F(\cdot)$ ,  $G(\cdot)$ , survival functions  $\bar{F}(\cdot)$ ,  $\bar{G}(\cdot)$ , probability density functions  $f(\cdot)$ ,  $g(\cdot)$ , hazard rate functions  $r_1(\cdot)$ ,  $r_2(\cdot)$ , and the reversed failure (hazard) rate functions  $\tilde{r}_1(\cdot)$  and  $\tilde{r}_2(\cdot)$ , respectively. Then

1.  $X$  is said to be smaller than  $Y$  in the

- (i) usual stochastic order (denoted as  $X \leq_{st} Y$ ) if  $\bar{F}(t) \leq \bar{G}(t)$  for all  $t$ ;
- (ii) failure (hazard) rate order (denoted as  $X \leq_{hr} Y$ ) if  $\bar{G}(t)/\bar{F}(t)$  is increasing in  $t \geq 0$ , or equivalently if  $r_1(t) \geq r_2(t)$  for all  $t \geq 0$ ;

- (iii) reversed failure (hazard) rate order (denoted as  $X \leq_{rhr} Y$ ) if  $G(t)/F(t)$  is increasing in  $t > 0$ , or equivalently if  $\tilde{r}_1(t) \leq \tilde{r}_2(t)$  for all  $t > 0$ ;
- (iv) likelihood ratio order (denoted as  $X \leq_{lr} Y$ ) if  $f(x)/g(x)$  decreases in  $x$  over the union of the supports of  $X$  and  $Y$ .

2.  $X$  is said to age faster than  $Y$  in terms of the

- (i) hazard rate (denoted as  $X \lesssim_{hr} Y$ ), if  $r_1(t)/r_2(t)$  is increasing in  $t > 0$ ;
- (ii) reversed hazard rate, denoted as  $X \lesssim_{rhr} Y$ , if  $\tilde{r}_2(t)/\tilde{r}_1(t)$  is increasing in  $t > 0$ .  $\square$

The following notation is used throughout the paper.

- (i)  $\mathcal{D} = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 \geq x_2 \geq \dots \geq x_n\}$ .
- (ii)  $\mathcal{D}_+ = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 \geq x_2 \geq \dots \geq x_n > 0\}$ .
- (iii)  $\mathcal{E} = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 \leq x_2 \leq \dots \leq x_n\}$ .
- (iv)  $\mathcal{E}_+ = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : 0 < x_1 \leq x_2 \leq \dots \leq x_n\}$ .

Before we start, we mention below, for completeness, a few lemmas to be used in sequel. The first four lemmas are due to Marshall et al. (2011). Below we take  $\mathbf{z} = (z_1, z_2, \dots, z_n)$  and  $\varphi_{(k)}(\mathbf{z}) = \partial\varphi(\mathbf{z})/\partial z_k$ , the partial derivative of  $\varphi$  with respect to its  $k$ th argument.

**Lemma 2.1** *Let  $\varphi : \mathcal{D} \rightarrow \mathbb{R}$  be a function, continuously differentiable on the interior of  $\mathcal{D}$ . Then, for  $\mathbf{x}, \mathbf{y} \in \mathcal{D}$ ,*

$$\mathbf{x} \stackrel{m}{\succeq} \mathbf{y} \implies \varphi(\mathbf{x}) \geq (\text{resp. } \leq) \varphi(\mathbf{y})$$

*if, and only if,*

$$\varphi_{(k)}(\mathbf{z}) \text{ is decreasing (resp. increasing) in } k = 1, 2, \dots, n.$$

**Lemma 2.2** *Let  $\varphi : \mathcal{E} \rightarrow \mathbb{R}$  be a function, continuously differentiable on the interior of  $\mathcal{E}$ . Then, for  $\mathbf{x}, \mathbf{y} \in \mathcal{E}$ ,*

$$\mathbf{x} \stackrel{m}{\succeq} \mathbf{y} \implies \varphi(\mathbf{x}) \geq (\text{resp. } \leq) \varphi(\mathbf{y})$$

*if, and only if,*

$$\varphi_{(k)}(\mathbf{z}) \text{ is increasing (resp. decreasing) in } k = 1, 2, \dots, n.$$

**Lemma 2.3** *Let  $I \subseteq \mathbb{R}^n$  be an open interval and let  $\varphi : I \rightarrow \mathbb{R}$  be continuously differentiable. Necessary and sufficient conditions for  $\varphi$  to be Schur-convex (resp. Schur-concave) on  $I^n$  are*

$\varphi$  is symmetric on  $I^n$ , and for all  $i \neq j$

$$(z_i - z_j) (\varphi_{(i)}(\mathbf{z}) - \varphi_{(j)}(\mathbf{z})) \geq (\text{resp. } \leq) 0 \text{ for all } \mathbf{z} \in I^n.$$

**Lemma 2.4** Let  $S \subseteq \mathbb{R}^n$ . Further, let  $\varphi : S \rightarrow \mathbb{R}$  be a function. Then, for  $\mathbf{x}, \mathbf{y} \in S$ ,

$$\mathbf{x} \succeq_w \mathbf{y} \implies \varphi(\mathbf{x}) \geq (\text{resp. } \leq) \varphi(\mathbf{y})$$

if, and only if,  $\varphi$  is both increasing (resp. decreasing) and Schur-convex (resp. Schur-concave) on  $S$ . Similarly,

$$\mathbf{x} \stackrel{w}{\preceq} \mathbf{y} \implies \varphi(\mathbf{x}) \geq (\text{resp. } \leq) \varphi(\mathbf{y})$$

if, and only if,  $\varphi$  is both decreasing (resp. increasing) and Schur-convex (resp. Schur-concave) on  $S$ .

The following lemma is taken from Khaledi and Kochar (2002) and Kundu et al. (2016).

**Lemma 2.5** Let  $\varphi : \mathbb{R}^{+n} \rightarrow \mathbb{R}$  be a function. Then,

$$\mathbf{x} \stackrel{p}{\preceq} \mathbf{y} \implies \varphi(\mathbf{x}) \geq (\text{resp. } \leq) \varphi(\mathbf{y})$$

if, and only if, the following two conditions hold:

- (i)  $\varphi(e^{a_1}, \dots, e^{a_n})$  is Schur-convex (resp. Schur-concave) in  $(a_1, \dots, a_n)$ ,
- (ii)  $\varphi(e^{a_1}, \dots, e^{a_n})$  is decreasing (resp. increasing) in each  $a_i$ , for  $i = 1, \dots, n$ ,

where  $a_i = \ln x_i$ , for  $i = 1, \dots, n$ .

Following lemma is adapted from Bon and Păltănea (2006) (See also Gupta et al., 2015).

**Lemma 2.6** Let  $\phi : (0, \infty)^n \rightarrow (0, \infty)$  be a symmetrical and continuously differentiable mapping. If, for  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in (0, \infty)^n$  with  $x_p = \min_{1 \leq i \leq n} x_i$  and  $x_q = \max_{1 \leq i \leq n} x_i$ , we have

$$(x_p - x_q) \left( \frac{1}{\prod_{i \neq p} x_i} \frac{\partial \phi}{\partial x_p} - \frac{1}{\prod_{i \neq q} x_i} \frac{\partial \phi}{\partial x_q} \right) < (>) 0,$$

for  $x_p \neq x_q$ , then

$$\phi(x_1, x_2, \dots, x_n) \leq (\geq) \phi(x, x, \dots, x),$$

where  $x = \sqrt[n]{x_1 x_2 \cdots x_n}$ .

### 3 Series systems with component lifetimes following PO model

In this section we compare the lifetimes of two series systems, each of heterogeneous components having lifetimes following the proportional odds (PO) model, with respect to some stochastic

orders. We also compare lifetimes of two series systems, one comprising of heterogeneous components and another comprising of homogeneous components.

Let  $X = (X_1, X_2, \dots, X_n)$  and  $Y = (Y_1, Y_2, \dots, Y_n)$  be two sets of independent random variables, each following PO model, denoted as  $X \sim PO(\bar{F}, \boldsymbol{\lambda})$  and  $Y \sim PO(\bar{F}, \boldsymbol{\mu})$ , where  $\bar{F}$  is the baseline survival function,  $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n)$  and  $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_n)$  with  $\lambda_i > 0$  and  $\mu_i > 0$ , for all  $i = 1, 2, \dots, n$ . We have the survival functions of  $X_{1:n}$  and  $Y_{1:n}$ , respectively, as

$$\bar{F}_{X_{1:n}}(x) = \prod_{i=1}^n \bar{F}_{X_i}(x) = \prod_{i=1}^n \frac{\lambda_i \bar{F}(x)}{1 - \bar{\lambda}_i \bar{F}(x)},$$

and

$$\bar{F}_{Y_{1:n}}(x) = \prod_{i=1}^n \bar{F}_{Y_i}(x) = \prod_{i=1}^n \frac{\mu_i \bar{F}(x)}{1 - \bar{\mu}_i \bar{F}(x)},$$

where  $\bar{\lambda}_i = 1 - \lambda_i$  and  $\bar{\mu}_i = 1 - \mu_i$ , for  $i = 1, 2, \dots, n$ .

The hazard rate functions of  $X_{1:n}$  and  $Y_{1:n}$  are, respectively, obtained as

$$r_{X_{1:n}}(x) = \sum_{i=1}^n r_{X_i}(x) = \sum_{i=1}^n \frac{r(x)}{1 - \bar{\lambda}_i \bar{F}(x)},$$

and

$$r_{Y_{1:n}}(x) = \sum_{i=1}^n r_{Y_i}(x) = \sum_{i=1}^n \frac{r(x)}{1 - \bar{\mu}_i \bar{F}(x)}.$$

If  $X \sim PO(\bar{F}, \lambda \mathbf{1})$ , where  $\mathbf{1} = (1, 1, \dots, 1)$ ,  $\lambda > 0$ , then the survival function and the hazard rate function of  $X_{1:n}$  are, respectively,

$$\bar{F}_{X_{1:n}}(x) = \frac{\lambda^n \bar{F}^n(x)}{(1 - \bar{\lambda} \bar{F}(x))^n},$$

and

$$r_{X_{1:n}}(x) = \frac{nr(x)}{1 - \bar{\lambda} \bar{F}(x)}.$$

The following theorem compares the lifetimes of two series systems formed out of  $n$  heterogeneous components following PO model.

**Theorem 3.1** *Suppose the lifetime vectors  $X \sim PO(\bar{F}, \boldsymbol{\lambda})$  and  $Y \sim PO(\bar{F}, \boldsymbol{\mu})$ . Then*

$$\boldsymbol{\lambda} \stackrel{p}{\succeq} \boldsymbol{\mu} \text{ implies } X_{1:n} \leq_{st} Y_{1:n}.$$

**Proof:** Write  $a_i = \ln \lambda_i$ ,  $i = 1, 2, \dots, n$ . Then

$$\begin{aligned}\bar{F}_{X_{1:n}}(x) &= \prod_{i=1}^n \frac{e^{a_i} \bar{F}(x)}{1 - (1 - e^{a_i}) \bar{F}(x)} \\ &= \phi(e^{a_1}, e^{a_2}, \dots, e^{a_n}), \text{ (say).}\end{aligned}$$

Note that  $\phi(e^{a_1}, e^{a_2}, \dots, e^{a_n})$  is symmetric with respect to  $(a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ . Now,

$$\frac{\partial \phi}{\partial a_i} = \frac{1 - \bar{F}(x)}{1 - (1 - e^{a_i}) \bar{F}(x)} \phi(e^{a_1}, e^{a_2}, \dots, e^{a_n}),$$

so that  $\phi(e^{a_1}, e^{a_2}, \dots, e^{a_n})$  is increasing in each  $a_i$ , for  $i = 1, 2, \dots, n$ . Now, for  $1 \leq i \leq j \leq n$ ,

$$\begin{aligned}(a_i - a_j) \left( \frac{\partial \phi}{\partial a_i} - \frac{\partial \phi}{\partial a_j} \right) &= \frac{(a_i - a_j)(e^{a_j} - e^{a_i}) \bar{F}(x)(1 - \bar{F}(x))}{(1 - (1 - e^{a_i}) \bar{F}(x))(1 - (1 - e^{a_j}) \bar{F}(x))} \phi(e^{a_1}, e^{a_2}, \dots, e^{a_n}) \\ &\leq 0.\end{aligned}$$

So, from Lemma 2.3,  $\phi(e^{a_1}, e^{a_2}, \dots, e^{a_n})$  is Schur-concave in  $(a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ . Thus, from Lemma 2.5, we have  $\phi(\lambda_1, \lambda_2, \dots, \lambda_n) \leq \phi(\mu_1, \mu_2, \dots, \mu_n)$  whenever  $\lambda \succeq_p \mu$ .  $\square$

**Corollary 3.1** Suppose that the lifetime vectors  $X \sim PO(\bar{F}, \lambda)$  and  $Y \sim PO(\bar{F}, \lambda_1)$ . Then,  $X_{1:n} \leq_{st} Y_{1:n}$  if  $\lambda \geq \sqrt[n]{\lambda_1 \lambda_2 \cdots \lambda_n}$ .  $\square$

The following counterexample shows that the condition of  $p$ -larger order given in the above theorem cannot be replaced by reciprocal majorization order.

**Counterexample 3.1** Let  $X = (X_1, X_2, X_3)$  and  $Y = (Y_1, Y_2, Y_3)$  be two sets of independent random variables, such that  $X_i \sim PO(\bar{F}, \lambda_i)$  and  $Y_i \sim PO(\bar{F}, \mu_i)$ ,  $i = 1, 2, 3$ , where the baseline survival function is given by  $\bar{F}(x) = e^{-2x}$ . Take  $(\lambda_1, \lambda_2, \lambda_3) = (2.2, 3, 5)$  and  $(\mu_1, \mu_2, \mu_3) = (2.8, 3.2, 3.3)$  so that  $(\lambda_1, \lambda_2, \lambda_3) \succeq_{rm} (\mu_1, \mu_2, \mu_3)$  but  $(\lambda_1, \lambda_2, \lambda_3) \not\prec_p (\mu_1, \mu_2, \mu_3)$ . It is observed that for  $x = 0.2$ ,  $\bar{F}_{X_{1:3}}(x) = 0.63929$  and  $\bar{F}_{Y_{1:3}}(x) = 0.641646$ . Again for  $x = 0.8$ ,  $\bar{F}_{X_{1:3}}(x) = 0.0861549$  and  $\bar{F}_{Y_{1:3}}(x) = 0.084394$ . So  $X_{1:3} \not\leq_{st} Y_{1:3}$ .

**Theorem 3.2** Suppose that the lifetime vectors  $X \sim PO(\bar{F}, \lambda)$  and  $Y \sim PO(\bar{F}, \mu)$ . Then

$$\lambda \succeq^w \mu \text{ implies } X_{1:n} \leq_{hr} Y_{1:n}.$$

**Proof:** We have

$$r_{X_{1:n}}(x) = \sum_{i=1}^n \frac{r(x)}{1 - \bar{\lambda}_i \bar{F}(x)},$$



which is symmetric with respect to  $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$ . Now,

$$\frac{\partial r_{X_{1:n}}(x)}{\partial \lambda_i} = -\frac{r(x)\bar{F}(x)}{(1 - \bar{\lambda}_i \bar{F}(x))^2},$$

so that  $r_{X_{1:n}}(x)$  is decreasing in each  $\lambda_i$ ,  $i = 1, 2, \dots, n$ . For  $1 \leq i \leq j \leq n$ ,

$$\begin{aligned} (\lambda_i - \lambda_j) \left( \frac{\partial r_{X_{1:n}}(x)}{\partial \lambda_i} - \frac{\partial r_{X_{1:n}}(x)}{\partial \lambda_j} \right) &= (\lambda_i - \lambda_j) r(x) \bar{F}(x) \left[ \frac{1}{(1 - \bar{\lambda}_j \bar{F}(x))^2} - \frac{1}{(1 - \bar{\lambda}_i \bar{F}(x))^2} \right] \\ &\stackrel{\text{sign}}{=} (\lambda_i - \lambda_j) [(1 - \bar{\lambda}_i \bar{F}(x))^2 - (1 - \bar{\lambda}_j \bar{F}(x))^2] \\ &\geq 0. \end{aligned}$$

So, from Lemma 2.3, it follows that  $r_{X_{1:n}}(x)$  is Schur-convex in  $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$ . Thus, by Lemma 2.4, we have  $r_{X_{1:n}}(x) \geq r_{Y_{1:n}}(x)$  whenever  $\boldsymbol{\lambda} \succeq^w \boldsymbol{\mu}$ . Hence the theorem follows.  $\square$

**Corollary 3.2** Suppose lifetime vectors  $X \sim PO(\bar{F}, \boldsymbol{\lambda})$  and  $Y \sim PO(\bar{F}, \boldsymbol{\lambda}1)$ . Then,  $X_{1:n} \leq_{hr} Y_{1:n}$  if  $\lambda \geq \frac{1}{n} \sum_{i=1}^n \lambda_i$ .  $\square$

The following counterexample shows that the condition of weakly supermojorization order given in the above theorem cannot be replaced by  $p$ -larger order.

**Counterexample 3.2** Let  $X = (X_1, X_2, X_3)$  and  $Y = (Y_1, Y_2, Y_3)$  are two sets of independent random variables, such that  $X_i \sim PO(\bar{F}, \lambda_i)$  and  $Y_i \sim PO(\bar{F}, \mu_i)$ ,  $i = 1, 2, 3$ , where the baseline survival function is given by  $\bar{F}(x) = e^{-1.2x}$ . Take  $(\lambda_1, \lambda_2, \lambda_3) = (2, 3, 5)$  and  $(\mu_1, \mu_2, \mu_3) = (2.8, 3.2, 3.4)$  so that  $(\lambda_1, \lambda_2, \lambda_3) \stackrel{p}{\succeq} (\mu_1, \mu_2, \mu_3)$  but  $(\lambda_1, \lambda_2, \lambda_3) \not\stackrel{w}{\succeq} (\mu_1, \mu_2, \mu_3)$ . It is observed that, for  $x = 0.2$ ,  $r_{X_{1:3}}(x) = 1.2297$  and  $r_{Y_{1:3}}(x) = 1.1687$ . Again, for  $x = 1.8$ ,  $r_{X_{1:3}}(x) = 2.3935$  and  $r_{Y_{1:3}}(x) = 2.4089$ . So  $X_{1:3} \not\leq_{hr} Y_{1:3}$ .  $\square$

In case of multiple-outlier model, below we study the relative ageing of two series systems with heterogeneous components in terms of the hazard rate.

**Theorem 3.3** Let  $X = (X_1, X_2, \dots, X_n)$  and  $Y = (Y_1, Y_2, \dots, Y_n)$  be two sets of independent random variables, each following the multiple-outlier PO model with  $X_i \sim PO(\bar{F}, \lambda_1)$ ,  $Y_i \sim PO(\bar{F}, \mu_1)$ , for  $i = 1, 2, \dots, n_1$ ,  $X_j \sim PO(\bar{F}, \lambda_2)$ ,  $Y_j \sim PO(\bar{F}, \mu_2)$ , for  $j = n_1 + 1, n_1 + 2, \dots, n_1 + n_2 (= n)$ . Then

$$\underbrace{(\lambda_1, \lambda_1, \dots, \lambda_1)}_{n_1} \underbrace{(\lambda_2, \lambda_2, \dots, \lambda_2)}_{n_2} \stackrel{m}{\succeq} \underbrace{(\mu_1, \mu_1, \dots, \mu_1)}_{n_1} \underbrace{(\mu_2, \mu_2, \dots, \mu_2)}_{n_2} \Rightarrow X_{1:n} \gtrsim_{hr} Y_{1:n},$$

provided  $\{(\lambda_1, \lambda_2) \in \mathcal{E}_+, (\mu_1, \mu_2) \in \mathcal{E}_+, n_1 \geq n_2\}$  or  $\{(\lambda_1, \lambda_2) \in \mathcal{D}_+, (\mu_1, \mu_2) \in \mathcal{D}_+, n_1 \leq n_2\}$ .

**Proof:** We denote

$$\mathcal{A} = \{\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n) : \lambda_i = \lambda_1 \text{ for } 1 \leq i \leq n_1 \text{ and } \lambda_j = \lambda_2, \text{ for } n_1 + 1 \leq j \leq n\}$$

and

$$\mathcal{B} = \{\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_n) : \mu_i = \mu_1 \text{ for } 1 \leq i \leq n_1 \text{ and } \mu_j = \mu_2, \text{ for } n_1 + 1 \leq j \leq n\}.$$

We have to show that, under the given majorization order,

$$\frac{r_{X_{1:n}}(x)}{r_{Y_{1:n}}(x)} = \frac{\sum_{i=1}^n \frac{1}{1-\lambda_i F(x)}}{\sum_{i=1}^n \frac{1}{1-\bar{\mu}_i F(x)}} \quad (3.1)$$

is decreasing in  $x > 0$  for  $\boldsymbol{\lambda} \in \mathcal{A}$ ,  $\boldsymbol{\mu} \in \mathcal{B}$ , which can be shown to be equivalent to

$$\frac{\sum_{i=1}^n \frac{\bar{\lambda}_i}{(1-\lambda_i F(x))^2}}{\sum_{i=1}^n \frac{1}{1-\lambda_i F(x)}} \geq \frac{\sum_{i=1}^n \frac{\bar{\mu}_i}{(1-\bar{\mu}_i F(x))^2}}{\sum_{i=1}^n \frac{1}{1-\bar{\mu}_i F(x)}}.$$

Now to show the above inequality, it suffices to show that, for  $\boldsymbol{\lambda} \in \mathcal{A}$  and  $\boldsymbol{\mu} \in \mathcal{B}$ ,

$$\phi(\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_n) =_{def} \frac{\sum_{i=1}^n \frac{\bar{\lambda}_i}{(1-\lambda_i F(x))^2}}{\sum_{i=1}^n \frac{1}{1-\lambda_i F(x)}}$$

is Schur-convex in  $(\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_n) \in \mathcal{A}$ . Writing  $u(x) = 1/(1-x)$  and  $v(x) = x/(1-x)$ , we have

$$\phi(\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_n) = \frac{1}{\bar{F}(x)} \frac{\sum_{i=1}^n u(\bar{\lambda}_i \bar{F}(x)) v(\bar{\lambda}_i \bar{F}(x))}{\sum_{i=1}^n u(\bar{\lambda}_i \bar{F}(x))}.$$

For  $1 \leq i \leq n_1$ ,

$$\frac{\partial \phi}{\partial \bar{\lambda}_i} = \frac{n_1 u(\bar{\lambda}_1 \bar{F}(x)) v'(\bar{\lambda}_1 \bar{F}(x)) [n_1 u(\bar{\lambda}_1 \bar{F}(x)) + n_2 u(\bar{\lambda}_2 \bar{F}(x))] + n_1 n_2 u(\bar{\lambda}_2 \bar{F}(x)) u'(\bar{\lambda}_1 \bar{F}(x)) [v(\bar{\lambda}_1 \bar{F}(x)) - v(\bar{\lambda}_2 \bar{F}(x))]}{(n_1 u(\bar{\lambda}_1 \bar{F}(x)) + n_2 u(\bar{\lambda}_2 \bar{F}(x)))^2},$$

and, for  $n_1 + 1 \leq j \leq n$ ,

$$\frac{\partial \phi}{\partial \bar{\lambda}_j} = \frac{n_2 u(\bar{\lambda}_2 \bar{F}(x)) v'(\bar{\lambda}_2 \bar{F}(x)) [n_1 u(\bar{\lambda}_1 \bar{F}(x)) + n_2 u(\bar{\lambda}_2 \bar{F}(x))] + n_1 n_2 u(\bar{\lambda}_1 \bar{F}(x)) u'(\bar{\lambda}_2 \bar{F}(x)) [v(\bar{\lambda}_2 \bar{F}(x)) - v(\bar{\lambda}_1 \bar{F}(x))]}{(n_1 u(\bar{\lambda}_1 \bar{F}(x)) + n_2 u(\bar{\lambda}_2 \bar{F}(x)))^2}.$$

Now, for  $1 \leq i, j \leq n_1$  or  $n_1 + 1 \leq i, j \leq n$ , we have  $\frac{\partial \phi}{\partial \bar{\lambda}_i} - \frac{\partial \phi}{\partial \bar{\lambda}_j} = 0$ . Again, for  $1 \leq i \leq n_1$  and  $n_1 + 1 \leq j \leq n$ , we have

$$\begin{aligned} \frac{\partial \phi}{\partial \bar{\lambda}_i} - \frac{\partial \phi}{\partial \bar{\lambda}_j} &=_{sign} [n_1 u(\bar{\lambda}_1 \bar{F}(x)) + n_2 u(\bar{\lambda}_2 \bar{F}(x))] [n_1 u(\bar{\lambda}_1 \bar{F}(x)) v'(\bar{\lambda}_1 \bar{F}(x)) - n_2 u(\bar{\lambda}_2 \bar{F}(x)) v'(\bar{\lambda}_2 \bar{F}(x))] \\ &\quad + n_1 n_2 [v(\bar{\lambda}_1 \bar{F}(x)) - v(\bar{\lambda}_2 \bar{F}(x))] [u(\bar{\lambda}_2 \bar{F}(x)) u'(\bar{\lambda}_1 \bar{F}(x)) + u(\bar{\lambda}_1 \bar{F}(x)) u'(\bar{\lambda}_2 \bar{F}(x))]. \end{aligned}$$

As  $v(x)$  and  $u(x)v'(x)$  are both increasing and nonnegative in  $x$ , we have, for  $n_1 \geq$  (resp.  $\leq$ )  $n_2$

and  $\bar{\lambda}_1 \geq$  (resp.  $\leq$ )  $\bar{\lambda}_2$ ,

$$\frac{\partial \phi}{\partial \bar{\lambda}_i} - \frac{\partial \phi}{\partial \bar{\lambda}_j} \geq \text{(resp. } \leq) 0.$$

So, from Lemma 2.1 and Lemma 2.2, we have

$$\underbrace{(\bar{\lambda}_1, \bar{\lambda}_1, \dots, \bar{\lambda}_1)}_{n_1} \underbrace{(\bar{\lambda}_2, \bar{\lambda}_2, \dots, \bar{\lambda}_2)}_{n_2} \succeq^m \underbrace{(\bar{\mu}_1, \bar{\mu}_1, \dots, \bar{\mu}_1)}_{n_1} \underbrace{(\bar{\mu}_2, \bar{\mu}_2, \dots, \bar{\mu}_2)}_{n_2} \Rightarrow X_{1:n} \succeq_{hr} Y_{1:n}.$$

Then the result follows from the fact that  $\underbrace{(\bar{\lambda}_1, \bar{\lambda}_1, \dots, \bar{\lambda}_1)}_{n_1} \underbrace{(\bar{\lambda}_2, \bar{\lambda}_2, \dots, \bar{\lambda}_2)}_{n_2} \succeq^m \underbrace{(\bar{\mu}_1, \bar{\mu}_1, \dots, \bar{\mu}_1)}_{n_1} \underbrace{(\bar{\mu}_2, \bar{\mu}_2, \dots, \bar{\mu}_2)}_{n_2}$

is equivalent to  $\underbrace{(\lambda_1, \lambda_1, \dots, \lambda_1)}_{n_1} \underbrace{(\lambda_2, \lambda_2, \dots, \lambda_2)}_{n_2} \succeq^m \underbrace{(\mu_1, \mu_1, \dots, \mu_1)}_{n_1} \underbrace{(\mu_2, \mu_2, \dots, \mu_2)}_{n_2}$ , which follows from Remark 2.1.

**Corollary 3.3** *Let, for  $i = 1, 2$ , the two independent random variables  $X_i$  and  $Y_i$  follow PO model with parameters  $\lambda_i$  and  $\mu_i$ , respectively. Then*

$$(\lambda_1, \lambda_2) \succeq^m (\mu_1, \mu_2) \Rightarrow X_{1:2} \succeq_{hr} Y_{1:2}.$$

Below we give another set of sufficient conditions for Theorem 3.3 to hold.

**Theorem 3.4** *Let  $X = (X_1, X_2, \dots, X_n)$  and  $Y = (Y_1, Y_2, \dots, Y_n)$  be two sets of independent random variables, each following the multiple-outlier PO model with  $X_i \sim PO(\bar{F}, \lambda_1)$ ,  $Y_i \sim PO(\bar{F}, \mu_1)$ , for  $i = 1, 2, \dots, n_1$ ,  $X_j \sim PO(\bar{F}, \lambda_2)$ ,  $Y_j \sim PO(\bar{F}, \mu_2)$ , for  $j = n_1 + 1, n_1 + 2, \dots, n_1 + n_2 (= n)$ . Then*

$$\max\{\lambda_1, \lambda_2\} \leq \min\{\mu_1, \mu_2\} \Rightarrow X_{1:n} \succeq_{hr} Y_{1:n}.$$

**Proof:** We have to show that

$$\frac{r_{X_{1:n}}(x)}{r_{Y_{1:n}}(x)} = \frac{\frac{n_1}{1-\lambda_1 \bar{F}(x)} + \frac{n_2}{1-\lambda_2 \bar{F}(x)}}{\frac{n_1}{1-\mu_1 \bar{F}(x)} + \frac{n_2}{1-\mu_2 \bar{F}(x)}} \text{ is decreasing in } x > 0. \quad (3.2)$$

Writing  $u(x) = 1/(1-x)$  and  $v(x) = x/(1-x)$ , (3.2) is equivalent to

$$\begin{aligned} & n_1^2 u(\bar{\mu}_1 \bar{F}(x)) u(\bar{\lambda}_1 \bar{F}(x)) [v(\bar{\mu}_1 \bar{F}(x)) - v(\bar{\lambda}_1 \bar{F}(x))] + n_1 n_2 u(\bar{\mu}_1 \bar{F}(x)) u(\bar{\lambda}_2 \bar{F}(x)) \\ & [v(\bar{\mu}_1 \bar{F}(x)) - v(\bar{\lambda}_2 \bar{F}(x))] + n_1 n_2 u(\bar{\mu}_2 \bar{F}(x)) u(\bar{\lambda}_1 \bar{F}(x)) [v(\bar{\mu}_2 \bar{F}(x)) - v(\bar{\lambda}_1 \bar{F}(x))] \\ & + n_2^2 u(\bar{\mu}_2 \bar{F}(x)) u(\bar{\lambda}_2 \bar{F}(x)) [v(\bar{\mu}_2 \bar{F}(x)) - v(\bar{\lambda}_2 \bar{F}(x))] \leq 0. \end{aligned}$$

As both  $u(x)$  and  $v(x)$  are increasing in  $x$ , so the above inequality holds if the condition  $\max\{\lambda_1, \lambda_2\} \leq \min\{\mu_1, \mu_2\}$  holds.  $\square$

A result on relative ageing is given next in terms of weakly majorization order.

**Theorem 3.5** Let  $X = (X_1, X_2, \dots, X_n)$  and  $Y = (Y_1, Y_2, \dots, Y_n)$  be two sets of independent random variables, each following the multiple-outlier PO model with  $X_i \sim PO(\bar{F}, \lambda_1)$ ,  $Y_i \sim PO(\bar{F}, \mu_1)$ , for  $i = 1, 2, \dots, n_1$ ,  $X_j \sim PO(\bar{F}, \eta)$ ,  $Y_j \sim PO(\bar{F}, \eta)$ , for  $j = n_1 + 1, n_1 + 2, \dots, n_1 + n_2 (= n)$ . Then

$$\underbrace{(\lambda_1, \lambda_1, \dots, \lambda_1)}_{n_1} \underbrace{(\eta, \eta, \dots, \eta)}_{n_2} \stackrel{w}{\succeq} \underbrace{(\mu_1, \mu_1, \dots, \mu_1)}_{n_1} \underbrace{(\eta, \eta, \dots, \eta)}_{n_2} \Rightarrow X_{1:n} \succ_{hr} Y_{1:n}.$$

**Proof:** We have to show that

$$\frac{r_{X_{1:n}}(x)}{r_{Y_{1:n}}(x)} = \frac{\frac{n_1}{1-\lambda_1 \bar{F}(x)} + \frac{n_2}{1-\eta \bar{F}(x)}}{\frac{n_1}{1-\mu_1 \bar{F}(x)} + \frac{n_2}{1-\eta \bar{F}(x)}} = \gamma(x), \text{ say,} \quad (3.3)$$

is decreasing in  $x > 0$ . As earlier, let us take  $u(x) = 1/(1-x)$  and  $v(x) = x/(1-x)$ , which are increasing in  $x$ . Now differentiating  $\gamma(x)$  with respect to  $x$ , we have

$$\begin{aligned} \gamma'(x) &=^{sign} n_1^2 u(\bar{\lambda}_1 \bar{F}(x)) u(\bar{\mu}_1 \bar{F}(x)) [v(\bar{\mu}_1 \bar{F}(x)) - v(\bar{\lambda}_1 \bar{F}(x))] + n_1 n_2 u(\bar{\lambda}_1 \bar{F}(x)) u(\bar{\eta} \bar{F}(x)) \\ &\quad [v(\bar{\eta} \bar{F}(x)) - v(\bar{\lambda}_1 \bar{F}(x))] + n_1 n_2 u(\bar{\eta} \bar{F}(x)) u(\bar{\mu}_1 \bar{F}(x)) [v(\bar{\mu}_1 \bar{F}(x)) - v(\bar{\eta} \bar{F}(x))] \\ &= \psi(x), \text{ say.} \end{aligned}$$

Now the condition  $\underbrace{(\lambda_1, \lambda_1, \dots, \lambda_1)}_{n_1} \underbrace{(\eta, \eta, \dots, \eta)}_{n_2} \stackrel{w}{\succeq} \underbrace{(\mu_1, \mu_1, \dots, \mu_1)}_{n_1} \underbrace{(\eta, \eta, \dots, \eta)}_{n_2}$  is equivalent to the fact that  $\lambda_1 \leq \eta \leq \mu_1$  or  $\lambda_1 \leq \mu_1 \leq \eta$  or  $\eta \leq \lambda_1 \leq \mu_1$ .

Case I: Let  $\lambda_1 \leq \eta \leq \mu_1$ . Then  $\psi(x) \leq 0$ .

Case II: Let  $\lambda_1 \leq \mu_1 \leq \eta$ . Then we have  $u(\bar{\lambda}_1 \bar{F}(x)) \geq u(\bar{\mu}_1 \bar{F}(x)) \geq u(\bar{\eta} \bar{F}(x))$  and  $v(\bar{\lambda}_1 \bar{F}(x)) \geq v(\bar{\mu}_1 \bar{F}(x)) \geq v(\bar{\eta} \bar{F}(x))$ , so that

$$\begin{aligned} \psi(x) &\leq n_1 u(\bar{\lambda}_1 \bar{F}(x)) [v(\bar{\mu}_1 \bar{F}(x)) - v(\bar{\lambda}_1 \bar{F}(x))] [n_1 u(\bar{\mu}_1 \bar{F}(x)) + n_2 u(\bar{\eta} \bar{F}(x))] \\ &\leq 0. \end{aligned}$$

Case III:  $\eta \leq \lambda_1 \leq \mu_1$ . Then the proof follows in the same line as that of Case II.

Hence the theorem follows.  $\square$

**Corollary 3.4** Let  $X_1$  and  $X_2$  be independent following PO model with parameters  $\lambda_1$  and  $\eta$  respectively, and let  $Y_1$  and  $Y_2$  be independent following PO model with parameters  $\mu_1$  and  $\eta$  respectively. Then

$$(\lambda_1, \eta) \stackrel{w}{\succeq} (\mu_1, \eta) \Rightarrow X_{1:2} \succ_{hr} Y_{1:2}.$$

The following lemma, required to prove the next theorem, has been borrowed from Kundu et al. (2016).

**Lemma 3.1** *If  $\lambda_1 \geq \mu_1 \geq \mu_2 \geq \lambda_2$  or  $\lambda_1 \leq \mu_1 \leq \mu_2 \leq \lambda_2$ , and  $n_1\lambda_1 + n_2\lambda_2 = n_1\mu_1 + n_2\mu_2$ , then*

$$\underbrace{(\lambda_1, \lambda_1, \dots, \lambda_1)}_{n_1} \underbrace{(\lambda_2, \lambda_2, \dots, \lambda_2)}_{n_2} \stackrel{m}{\succeq} \underbrace{(\mu_1, \mu_1, \dots, \mu_1)}_{n_1} \underbrace{(\mu_2, \mu_2, \dots, \mu_2)}_{n_2}.$$

The following theorem shows that under certain restriction on the model parameters the condition of majorization order in Theorem 3.3 can be replaced by the weak supermajorization order.

**Theorem 3.6** *Let  $X = (X_1, X_2, \dots, X_n)$  and  $Y = (Y_1, Y_2, \dots, Y_n)$  be two sets of independent random variables, each following the multiple-outlier PO model with  $X_i \sim PO(\bar{F}, \lambda_1)$ ,  $Y_i \sim PO(\bar{F}, \mu_1)$ , for  $i = 1, 2, \dots, n_1$ ,  $X_j \sim PO(\bar{F}, \lambda_2)$ ,  $Y_j \sim PO(\bar{F}, \mu_2)$ , for  $j = n_1 + 1, n_1 + 2, \dots, n_1 + n_2 (= n)$ . Then, for  $\{\lambda_1 \leq \mu_1 \leq \mu_2 \leq \lambda_2 \text{ and } n_1 \geq n_2\}$  or  $\{\lambda_1 \geq \mu_1 \geq \mu_2 \geq \lambda_2 \text{ and } n_1 \leq n_2\}$*

$$\underbrace{(\lambda_1, \lambda_1, \dots, \lambda_1)}_{n_1} \underbrace{(\lambda_2, \lambda_2, \dots, \lambda_2)}_{n_2} \stackrel{w}{\succeq} \underbrace{(\mu_1, \mu_1, \dots, \mu_1)}_{n_1} \underbrace{(\mu_2, \mu_2, \dots, \mu_2)}_{n_2} \Rightarrow X_{1:n} \succeq_{hr} Y_{1:n}.$$

**Proof:** Suppose that the first set of conditions holds. The weak supermajorization order gives that  $\lambda_1 \leq \mu_1$  and  $n_1\lambda_1 + r\lambda_2 \leq n_1\mu_1 + r\mu_2$ , for  $r = 1, 2, \dots, n_2$ . If  $n_1\lambda_1 + n_2\lambda_2 = n_1\mu_1 + n_2\mu_2$  holds then, under the given condition, the result follows from Theorem 3.3. Suppose that  $n_1\lambda_1 + n_2\lambda_2 < n_1\mu_1 + n_2\mu_2$ . Then there exists an  $\eta$  satisfying  $\lambda_1 < \eta \leq \mu_1$  such that  $n_1\eta + n_2\lambda_2 = n_1\mu_1 + n_2\mu_2$ . Let  $X_{1:n}^*$  be the lifetime of a series system formed by  $n$  components having lifetimes  $X_1^*, X_2^*, \dots, X_n^*$ , where  $X_i^* \sim PO(\bar{F}, \eta)$ , for  $i = 1, 2, \dots, n_1$  and  $X_j^* \sim PO(\bar{F}, \lambda_2)$ , for  $j = n_1 + 1, n_1 + 2, \dots, n_1 + n_2 (= n)$ . Then, from Lemma 3.1 and Theorem 3.3, we have  $X_{1:n}^* \succeq_{hr} Y_{1:n}$ , when  $n_1 \geq n_2$ . Again  $\lambda_1 < \eta \leq \lambda_2$  and

$$\underbrace{(\lambda_1, \lambda_1, \dots, \lambda_1)}_{n_1} \underbrace{(\lambda_2, \lambda_2, \dots, \lambda_2)}_{n_2} \stackrel{w}{\succeq} \underbrace{(\eta, \eta, \dots, \eta)}_{n_1} \underbrace{(\lambda_2, \lambda_2, \dots, \lambda_2)}_{n_2}.$$

So, from Theorem 3.5, it follows that  $X_{1:n} \succeq_{hr} X_{1:n}^*$ . Hence  $X_{1:n} \succeq_{hr} Y_{1:n}$ . The proof for the second set of conditions can be done in a similar way.  $\square$

**Corollary 3.5** *Let  $X_1$  and  $X_2$  be independent following PO model with parameters  $\lambda_1$  and  $\lambda_2$  respectively, and let  $Y_1$  and  $Y_2$  be independent following PO model with parameters  $\mu_1$  and  $\mu_2$  respectively. Then*

$$(\lambda_1, \lambda_2) \stackrel{w}{\succeq} (\mu_1, \mu_2) \Rightarrow X_{1:2} \succeq_{hr} Y_{1:2},$$

where  $\{\lambda_1 \leq \mu_1 \leq \mu_2 \leq \lambda_2\}$  or  $\{\lambda_1 \geq \mu_1 \geq \mu_2 \geq \lambda_2\}$ .  $\square$

The following theorem shows that, under certain condition, a series system with homogeneous components ages faster than that with heterogeneous ones in terms of the hazard rate.

**Theorem 3.7** Suppose lifetime vectors  $X \sim PO(\bar{F}, \lambda)$  and  $Y \sim PO(\bar{F}, \lambda 1)$ . Then,  $X_{1:n} \gtrsim_{hr} Y_{1:n}$  if  $\lambda \geq \frac{1}{n} \sum_{i=1}^n \lambda_i$ .

**Proof:** We have

$$\frac{r_{X_{1:n}}(x)}{r_{Y_{1:n}}(x)} = \frac{1 - \bar{\lambda}\bar{F}(x)}{n} \sum_{i=1}^n \frac{1}{1 - \bar{\lambda}_i\bar{F}(x)}.$$

Now, differentiating the above expression with respect to  $x$ , we have, for  $x > 0$ ,

$$\frac{d}{dx} \left( \frac{r_{X_{1:n}}(x)}{r_{Y_{1:n}}(x)} \right) = \frac{f(x)(1 - \bar{\lambda}\bar{F}(x))}{n} \left[ \left( \frac{\bar{\lambda}}{1 - \bar{\lambda}\bar{F}(x)} \right) \left( \sum_{i=1}^n \frac{1}{1 - \bar{\lambda}_i\bar{F}(x)} \right) - \sum_{i=1}^n \frac{\bar{\lambda}_i}{(1 - \bar{\lambda}_i\bar{F}(x))^2} \right],$$

so that  $\frac{r_{X_{1:n}}(x)}{r_{Y_{1:n}}(x)}$  is decreasing if

$$\left( \frac{\bar{\lambda}\bar{F}(x)}{1 - \bar{\lambda}\bar{F}(x)} \right) \left( \sum_{i=1}^n \frac{1}{1 - \bar{\lambda}_i\bar{F}(x)} \right) \leq \sum_{i=1}^n \frac{\bar{\lambda}_i\bar{F}(x)}{(1 - \bar{\lambda}_i\bar{F}(x))^2}.$$

From Cebyšev's inequality (cf. Mitrinović et al., 1993, p. 240), the above inequality holds if

$$\left( \frac{\bar{\lambda}\bar{F}(x)}{1 - \bar{\lambda}\bar{F}(x)} \right) \left( \sum_{i=1}^n \frac{1}{1 - \bar{\lambda}_i\bar{F}(x)} \right) \leq \frac{1}{n} \left( \sum_{i=1}^n \frac{\bar{\lambda}_i\bar{F}(x)}{1 - \bar{\lambda}_i\bar{F}(x)} \right) \left( \sum_{i=1}^n \frac{1}{1 - \bar{\lambda}_i\bar{F}(x)} \right)$$

or equivalently,

$$\frac{\bar{\lambda}\bar{F}(x)}{1 - \bar{\lambda}\bar{F}(x)} \leq \frac{1}{n} \sum_{i=1}^n \frac{\bar{\lambda}_i\bar{F}(x)}{1 - \bar{\lambda}_i\bar{F}(x)}. \quad (3.4)$$

Let  $\phi(x) = x/(1 - x)$ , which is increasing and convex in  $x$ . Now (3.4) holds if

$$\phi(\bar{\lambda}\bar{F}(x)) \leq \frac{1}{n} \sum_{i=1}^n \phi(\bar{\lambda}_i\bar{F}(x)),$$

i.e. if

$$\phi(\bar{\lambda}\bar{F}(x)) \leq \phi \left( \frac{1}{n} \sum_{i=1}^n \bar{\lambda}_i\bar{F}(x) \right),$$

which follows from the fact that  $\phi$  is convex. Now the theorem holds because  $\phi$  is increasing.  $\square$

In case of multiple-outlier model, below we study the likelihood ratio ordering between two series systems with heterogeneous components. The result under majorization order follows from Theorems 3.2 and 3.3, whereas the result under weak supermajorization order follows from Theorems 3.2 and 3.6.

**Theorem 3.8** Let  $X = (X_1, X_2, \dots, X_n)$  and  $Y = (Y_1, Y_2, \dots, Y_n)$  be two sets of independent random variables, each following the multiple-outlier PO model such that  $X_i \sim PO(\bar{F}, \lambda_1)$ ,  $Y_i \sim PO(\bar{F}, \mu_1)$ , for  $i = 1, 2, \dots, n_1$ ,  $X_j \sim PO(\bar{F}, \lambda_2)$ ,  $Y_j \sim PO(\bar{F}, \mu_2)$ , for  $j = n_1 + 1, n_1 +$

$2, \dots, n_1 + n_2 (= n)$ . Then

$$\underbrace{(\lambda_1, \lambda_1, \dots, \lambda_1)}_{n_1} \underbrace{(\lambda_2, \lambda_2, \dots, \lambda_2)}_{n_2} \stackrel{m}{\succeq} (\text{resp. } \stackrel{w}{\succeq}) \underbrace{(\mu_1, \mu_1, \dots, \mu_1)}_{n_1} \underbrace{(\mu_2, \mu_2, \dots, \mu_2)}_{n_2} \Rightarrow X_{1:n} \leq_{lr} Y_{1:n},$$

provided  $\{(\lambda_1, \lambda_2) \in \mathcal{E}_+, (\mu_1, \mu_2) \in \mathcal{E}_+, n_1 \geq n_2\}$  or  $\{(\lambda_1, \lambda_2) \in \mathcal{D}_+, (\mu_1, \mu_2) \in \mathcal{D}_+, n_1 \leq n_2\}$  [resp.  $\{\lambda_1 \leq \mu_1 \leq \mu_2 \leq \lambda_2 \text{ and } n_1 \geq n_2\}$  or  $\{\lambda_1 \geq \mu_1 \geq \mu_2 \geq \lambda_2 \text{ and } n_1 \leq n_2\}$ ] holds.  $\square$

The following theorem gives a condition under which a series system with homogeneous components and that with heterogeneous ones are ordered in terms of the likelihood ratio order. The proof follows from Theorem 3.7 and Corollary 3.2.

**Theorem 3.9** Suppose lifetime vectors  $X \sim PO(\bar{F}, \boldsymbol{\lambda})$  and  $Y \sim PO(\bar{F}, \boldsymbol{\lambda}1)$ . Then,  $X_{1:n} \leq_{lr} Y_{1:n}$  if  $\lambda \geq \frac{1}{n} \sum_{i=1}^n \lambda_i$ .

## 4 Parallel systems with component lifetimes following PO model

In this section we compare lifetimes of two parallel systems of heterogeneous components having lifetimes following the PO model with respect to some stochastic orders. We also compare lifetimes of two parallel systems, one comprising of heterogeneous components and another of homogeneous components.

Let  $X = (X_1, X_2, \dots, X_n)$  and  $Y = (Y_1, Y_2, \dots, Y_n)$  be two sets of independent random variables, each following PO model. Let  $X \sim PO(\bar{F}, \boldsymbol{\lambda})$  and  $Y \sim PO(\bar{F}, \boldsymbol{\mu})$ , where  $\bar{F}$  is the baseline survival function,  $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n)$  and  $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_n)$ ,  $\lambda_i > 0$  and  $\mu_i > 0$ ,  $i = 1, 2, \dots, n$ . We have the survival functions of  $X_{n:n}$  and  $Y_{n:n}$ , respectively, as

$$\bar{F}_{X_{n:n}}(x) = 1 - \prod_{i=1}^n (1 - \bar{F}_{X_i}(x)) = 1 - \prod_{i=1}^n \left( \frac{1 - \bar{F}(x)}{1 - \bar{\lambda}_i \bar{F}(x)} \right), \quad (4.1)$$

and

$$\bar{F}_{Y_{n:n}}(x) = 1 - \prod_{i=1}^n (1 - \bar{F}_{Y_i}(x)) = 1 - \prod_{i=1}^n \left( \frac{1 - \bar{F}(x)}{1 - \bar{\mu}_i \bar{F}(x)} \right).$$

Also the reversed hazard rate functions of  $X_{n:n}$  and  $Y_{n:n}$  are obtained, respectively, as

$$\tilde{r}_{X_{n:n}}(x) = \sum_{i=1}^n \tilde{r}_{X_i} = \sum_{i=1}^n \frac{\lambda_i \tilde{r}(x)}{1 - \bar{\lambda}_i \bar{F}(x)}, \quad (4.2)$$

and

$$\tilde{r}_{Y_{n:n}}(x) = \sum_{i=1}^n \tilde{r}_{Y_i} = \sum_{i=1}^n \frac{\mu_i \tilde{r}(x)}{1 - \bar{\mu}_i \bar{F}(x)}.$$

If  $X \sim PO(\bar{F}, \boldsymbol{\lambda}1)$ ,  $\lambda > 0$ , then the survival function and the reversed hazard rate function of

$X_{n:n}$  are, respectively, given by

$$\bar{F}_{X_{n:n}}(x) = 1 - \left( \frac{1 - \bar{F}(x)}{1 - \bar{\lambda}\bar{F}(x)} \right)^n,$$

and

$$\tilde{r}_{X_{n:n}}(x) = \frac{n\lambda\tilde{r}(x)}{1 - \bar{\lambda}\bar{F}(x)}.$$

The following theorem compares the lifetimes of two parallel systems formed out of  $n$  heterogeneous components following PO model.

**Theorem 4.1** *Suppose that lifetime vectors  $X \sim PO(\bar{F}, \boldsymbol{\lambda})$  and  $Y \sim PO(\bar{F}, \boldsymbol{\mu})$ . Then*

$$\boldsymbol{\lambda} \stackrel{w}{\succeq} \boldsymbol{\mu} \text{ implies } X_{n:n} \leq_{rhr} Y_{n:n}.$$

**Proof:** Differentiating (4.2) with respect to  $\lambda_i$  we have

$$\begin{aligned} \frac{\partial \tilde{r}_{X_{n:n}}}{\partial \lambda_i} &= \frac{\tilde{r}(x)(1 - \bar{F}(x))}{(1 - \bar{\lambda}_i \bar{F}(x))^2} \\ &\geq 0, \end{aligned}$$

so that  $\tilde{r}_{X_{n:n}}(x)$  is increasing in each  $\lambda_i$ ,  $i = 1, 2, \dots, n$ . Also  $\tilde{r}_{X_{n:n}}(x)$  is symmetric with respect to  $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$ . For  $1 \leq i \leq j \leq n$ ,

$$\begin{aligned} (\lambda_i - \lambda_j) \left( \frac{\partial \tilde{r}_{X_{n:n}}}{\partial \lambda_i} - \frac{\partial \tilde{r}_{X_{n:n}}}{\partial \lambda_j} \right) &= (\lambda_i - \lambda_j) \tilde{r}(x)(1 - \bar{F}(x)) \left[ \frac{1}{(1 - \bar{\lambda}_i \bar{F}(x))^2} - \frac{1}{(1 - \bar{\lambda}_j \bar{F}(x))^2} \right] \\ &\stackrel{sign}{=} (\lambda_i - \lambda_j) ((1 - \bar{\lambda}_j \bar{F}(x))^2 - (1 - \bar{\lambda}_i \bar{F}(x))^2) \\ &\leq 0. \end{aligned}$$

So, from Lemma 2.3, it follows that  $\tilde{r}_{X_{n:n}}(x)$  is Schur-concave in  $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$ . Thus, from Lemma 2.4, we have  $\tilde{r}_{X_{n:n}}(x) \leq \tilde{r}_{Y_{n:n}}(x)$  whenever  $\boldsymbol{\lambda} \stackrel{w}{\succeq} \boldsymbol{\mu}$ .  $\square$

**Corollary 4.1** *Suppose lifetime vectors  $X \sim PO(\bar{F}, \boldsymbol{\lambda})$  and  $Y \sim PO(\bar{F}, \lambda \mathbf{1})$ . Then,  $X_{n:n} \leq_{rhr} Y_{n:n}$  if  $\lambda \geq \frac{1}{n} \sum_{i=1}^n \lambda_i$ .*  $\square$

The following counterexample shows that even under usual stochastic order, the condition of weakly supermojorization order given in the above theorem cannot be replaced by  $p$ -larger order.

**Counterexample 4.1** *Let  $X = (X_1, X_2, X_3)$  and  $Y = (Y_1, Y_2, Y_3)$  be two sets of independent random variables, such that  $X_i \sim PO(\bar{F}, \lambda_i)$  and  $Y_i \sim PO(\bar{F}, \mu_i)$ ,  $i = 1, 2, 3$ , where the baseline survival function is given by  $\bar{F}(x) = e^{-1.8x}$ ,  $x > 0$ . Take  $(\lambda_1, \lambda_2, \lambda_3) = (2, 3, 5)$  and  $(\mu_1, \mu_2, \mu_3) = (2.6, 3.2, 3.7)$  so that  $(\lambda_1, \lambda_2, \lambda_3) \stackrel{p}{\succeq} (\mu_1, \mu_2, \mu_3)$  but  $(\lambda_1, \lambda_2, \lambda_3) \not\stackrel{w}{\succeq} (\mu_1, \mu_2, \mu_3)$ . It is*



observed that, for  $x = 1.5$ ,  $\bar{F}_{X_{3:3}}(x) = 0.471629$  and  $\bar{F}_{Y_{3:3}}(x) = 0.459619$ . So  $X_{3:3} \not\geq_{st} Y_{3:3}$ . Now take  $(\lambda_1, \lambda_2, \lambda_3) = (2.5, 3, 5)$  and  $(\mu_1, \mu_2, \mu_3) = (3, 3.8, 4.4)$  so that  $(\lambda_1, \lambda_2, \lambda_3) \stackrel{p}{\succeq} (\mu_1, \mu_2, \mu_3)$ . It is observed that, for  $x = 1.2$ ,  $\bar{F}_{X_{3:3}}(x) = 0.67176$  and  $\bar{F}_{Y_{3:3}}(x) = 0.69449$ . So  $X_{3:3} \not\geq_{st} Y_{3:3}$ .  $\square$

**Theorem 4.2** Suppose that lifetime vectors  $X \sim PO(\bar{F}, \lambda)$  and  $Y \sim PO(\bar{F}, \lambda 1)$ . Then,  $X_{n:n} \geq_{st} Y_{n:n}$  if  $\lambda = \sqrt[n]{\lambda_1 \lambda_2 \cdots \lambda_n}$ .

**Proof:** Write

$$\bar{F}_{X_{n:n}}(x) = \phi(\lambda_1, \lambda_2, \dots, \lambda_n)$$

Then we have

$$\frac{\partial \phi}{\partial \lambda_i} = \bar{F}(x)[1 - \bar{F}_{X_{n:n}}(x)] \frac{1}{1 - \bar{\lambda}_i \bar{F}(x)}.$$

Let  $\lambda_p = \min_{1 \leq i \leq n} \lambda_i$  and  $\lambda_q = \max_{1 \leq i \leq n} \lambda_i$ . Then

$$\begin{aligned} \left( \frac{1}{\prod_{i \neq p} \lambda_i} \right) \frac{\partial \phi}{\partial \lambda_p} - \left( \frac{1}{\prod_{i \neq q} \lambda_i} \right) \frac{\partial \phi}{\partial \lambda_q} &=_{sign} \left( \frac{1}{\prod_{i \neq p} \lambda_i} \right) \frac{1}{1 - \bar{\lambda}_p \bar{F}(x)} - \left( \frac{1}{\prod_{i \neq q} \lambda_i} \right) \frac{1}{1 - \bar{\lambda}_q \bar{F}(x)} \\ &=_{sign} \left[ \frac{\lambda_p}{1 - \bar{\lambda}_p \bar{F}(x)} - \frac{\lambda_q}{1 - \bar{\lambda}_q \bar{F}(x)} \right] \\ &=_{sign} (\lambda_p - \lambda_q)(1 - \bar{F}(x)) \\ &< 0. \end{aligned}$$

So  $(\lambda_p - \lambda_q) \left( \frac{1}{\prod_{i \neq p} \lambda_i} \frac{\partial \phi}{\partial \lambda_p} - \frac{1}{\prod_{i \neq q} \lambda_i} \frac{\partial \phi}{\partial \lambda_q} \right) > 0$ . Thus, from Lemma 2.6, we have, for  $\lambda = \sqrt[n]{\lambda_1 \lambda_2 \cdots \lambda_n}$ ,  $\phi(\lambda_1, \lambda_2, \dots, \lambda_n) \geq \phi(\lambda, \lambda, \dots, \lambda)$ , i.e.  $X_{n:n} \geq_{st} Y_{n:n}$ .  $\square$

Following counterexample shows that even in case of multiple-outlier model, under the majorization order, two parallel systems of heterogeneous components may not be ordered with respect to relative ageing in terms of reversed hazard rate.

**Counterexample 4.2** Let  $X = (X_1, X_2, \dots, X_6)$  and  $Y = (Y_1, Y_2, \dots, Y_6)$  be two sets of independent random variables, each following the multiple-outlier PO model such that  $X_i \sim PO(\bar{F}, 2)$ ,  $Y_i \sim PO(\bar{F}, 3)$ , for  $i = 1, 2$ ,  $X_j \sim PO(\bar{F}, 6)$ ,  $Y_j \sim PO(\bar{F}, 5.5)$ , for  $j = 3, 4, 5, 6$ , where the baseline survival function is given by  $\bar{F}(x) = e^{-2x}$ . Clearly,  $(2, 2, 6, 6, 6, 6) \stackrel{m}{\succeq} (3, 3, 5.5, 5.5, 5.5, 5.5)$ . However, it is observed from Figure 1(a) that  $\tilde{r}_{Y_{6:6}}(x)/\tilde{r}_{X_{6:6}}(x)$  is nonmonotone.

**Remark 4.1** Taking the random variables as in Counterexample 4.2, we see from Figure 1(b) that  $f_{Y_{6:6}}(x)/f_{X_{6:6}}(x)$  is also nonmonotone. This gives that, in case of multiple-outlier model, under the majorization order, two parallel systems with heterogeneous components may not be ordered with respect to likelihood ratio order.

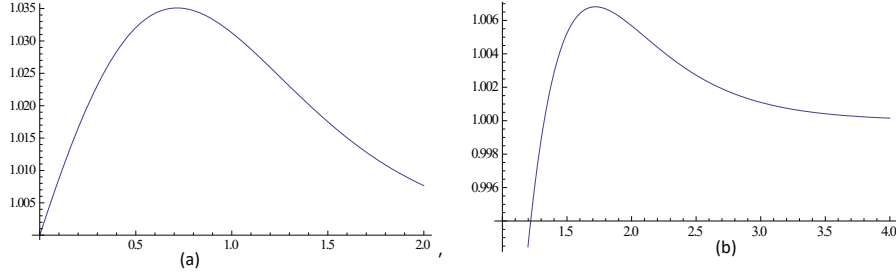


Figure 1: (a) Curve  $\tilde{r}_{Y_{6:6}}(x)/\tilde{r}_{X_{6:6}}(x)$  (b) Curve  $f_{Y_{6:6}}(x)/f_{X_{6:6}}(x)$  for  $x > 0$ .

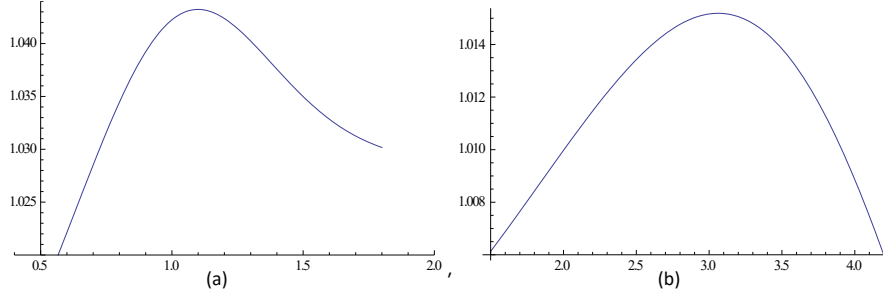


Figure 2: (a) Curve  $\tilde{r}_{Y_{4:4}}(x)/\tilde{r}_{X_{4:4}}(x)$  (b) Curve  $\tilde{r}_{Y_{4:4}}(x)/\tilde{r}_{X_{4:4}}(x)$  for  $x > 0$ .

Following counterexample shows that, a parallel system of heterogeneous components may not be comparable with that of homogeneous components with respect to relative ageing in terms of reversed hazard rate, irrespective of the condition in Corollary 4.1.

**Counterexample 4.3** Let  $X = (X_1, X_2, X_3, X_4)$  and  $Y = (Y_1, Y_2, Y_3, Y_4)$  be two sets of independent random variables, each following the multiple-outlier PO model such that  $X_i \sim PO(\bar{F}, \lambda_i)$ ,  $i = 1, 2, 3, 4$  and  $Y_i \sim PO(\bar{F}, \lambda)$ ,  $i = 1, 2, 3, 4$ , where the baseline survival function is given by  $\bar{F}(x) = e^{-(x/\beta)^k}$ ,  $\beta, k > 0$ . It is observed from Figure 2(a) that for  $\lambda_1 = 2$ ,  $\lambda_2 = 3$ ,  $\lambda_3 = 4$ ,  $\lambda_4 = 5$ ,  $\lambda = 3.6$ ,  $\beta = 0.8$ , and  $k = 2$   $\tilde{r}_{Y_{4:4}}(x)/\tilde{r}_{X_{4:4}}(x)$  is nonmonotone. Again for  $\lambda_1 = 2$ ,  $\lambda_2 = 3$ ,  $\lambda_3 = 4$ ,  $\lambda_4 = 5$ ,  $\lambda = 3.4$ ,  $\beta = 3$  and  $k = 2$   $\tilde{r}_{Y_{4:4}}(x)/\tilde{r}_{X_{4:4}}(x)$  is also nonmonotone as seen from Figure 2(b).  $\square$

In case of multiple-outlier model, following theorem gives a condition under which  $X_{n:n}$  ages faster than  $Y_{n:n}$  in terms of the reversed hazard rate.

**Theorem 4.3** Let  $X = (X_1, X_2, \dots, X_n)$  and  $Y = (Y_1, Y_2, \dots, Y_n)$  be two sets of independent random variables, each following the multiple-outlier PO model such that  $X_i \sim PO(\bar{F}, \lambda_1)$ ,  $Y_i \sim PO(\bar{F}, \mu_1)$ , for  $i = 1, 2, \dots, n_1$ ,  $X_j \sim PO(\bar{F}, \eta)$ ,  $Y_j \sim PO(\bar{F}, \eta)$ , for  $j = n_1 + 1, n_1 + 2, \dots, n_1 + n_2 (= n)$ . Then

$$\lambda_1 \leq \eta \leq \mu_1 \Rightarrow X_{n:n} \lesssim_{rhr} Y_{n:n}.$$

**Proof:** We have to show that

$$\frac{\tilde{r}_{Y_{n:n}}(x)}{\tilde{r}_{X_{n:n}}(x)} = \frac{\frac{n_1\mu_1}{1-\bar{\mu}_1\bar{F}(x)} + \frac{n_2\eta}{1-\bar{\eta}\bar{F}(x)}}{\frac{n_1\lambda_1}{1-\bar{\lambda}_1\bar{F}(x)} + \frac{n_2\eta}{1-\bar{\eta}\bar{F}(x)}} = \gamma(x), \text{ say,} \quad (4.3)$$

is increasing in  $x > 0$ . Let us write  $u(x) = 1/(1-x)$  and  $v(x) = x/(1-x)$ , both of which are increasing in  $x$ . Now differentiating  $\gamma(x)$  with respect to  $x$ , we have

$$\begin{aligned} \gamma'(x) &=^{sign} n_1^2\lambda_1\mu_1 u(\bar{\lambda}_1\bar{F}(x))u(\bar{\mu}_1\bar{F}(x))[v(\bar{\lambda}_1\bar{F}(x)) - v(\bar{\mu}_1\bar{F}(x))] + n_1n_2\eta\mu_1 u(\bar{\mu}_1\bar{F}(x))u(\bar{\eta}\bar{F}(x)) \\ &\quad [v(\bar{\eta}\bar{F}(x)) - v(\bar{\mu}_1\bar{F}(x))] + n_1n_2\eta\lambda_1 u(\bar{\lambda}_1\bar{F}(x))u(\bar{\eta}\bar{F}(x))[v(\bar{\lambda}_1\bar{F}(x)) - v(\bar{\eta}\bar{F}(x))] \\ &\geq 0, \end{aligned}$$

if  $\lambda_1 \leq \eta \leq \mu_1$ . Hence the theorem follows.

**Corollary 4.2** *Let  $X_1$  and  $X_2$  be independent following PO model with parameters  $\lambda_1$  and  $\eta$  respectively, and let  $Y_1$  and  $Y_2$  be independent following PO model with parameters  $\mu_1$  and  $\eta$  respectively. Then*

$$\lambda_1 \leq \eta \leq \mu_1 \Rightarrow X_{2:2} \lesssim_{rhr} Y_{2:2}.$$

The following counterexample shows that Theorem 4.3 does not hold under the condition  $\lambda_1 \leq \mu_1 \leq \eta$ .

**Counterexample 4.4** *Let  $X_1$  and  $X_2$  follow PO model with parameters  $\lambda_1$  and  $\eta$  respectively, and  $Y_1$  and  $Y_2$  follow PO model with parameters  $\mu_1$  and  $\eta$  respectively, where the baseline distribution is exponential with parameter  $\lambda = 2$ . Now for  $\lambda_1 = 0.2$ ,  $\mu_1 = 0.4$  and  $\eta = 0.9$ ,  $\tilde{r}_{Y_{2:2}}(x)/\tilde{r}_{X_{2:2}}(x)$  is nonmonotone as seen from Figure 3.*

**Theorem 4.4** *Let  $X = (X_1, X_2, \dots, X_n)$  and  $Y = (Y_1, Y_2, \dots, Y_n)$  be two sets of independent random variables, each following the multiple-outlier PO model such that  $X_i \sim PO(\bar{F}, \lambda_1)$ ,  $Y_i \sim PO(\bar{F}, \mu_1)$ , for  $i = 1, 2, \dots, n_1$ ,  $X_j \sim PO(\bar{F}, \eta)$ ,  $Y_j \sim PO(\bar{F}, \eta)$ , for  $j = n_1 + 1, n_1 + 2, \dots, n_1 + n_2 (= n)$ . Then*

$$\lambda_1 \leq \eta \leq \mu_1 \Rightarrow X_{n:n} \leq_{lr} Y_{n:n}.$$

**Proof:** We have to show that

$$\frac{f_{Y_{n:n}}(x)}{f_{X_{n:n}}(x)} = \frac{F_{Y_{n:n}}(x)}{F_{X_{n:n}}(x)} \frac{\tilde{r}_{Y_{n:n}}(x)}{\tilde{r}_{X_{n:n}}(x)} \quad (4.4)$$

is increasing in  $x > 0$ . We have  $\lambda_1 \leq \eta \leq \mu_1$ , which implies  $\underbrace{(\lambda_1, \lambda_1, \dots, \lambda_1)}_{n_1}, \underbrace{(\eta, \eta, \dots, \eta)}_{n_2} \stackrel{w}{\succeq} \underbrace{(\mu_1, \mu_1, \dots, \mu_1)}_{n_1}, \underbrace{(\eta, \eta, \dots, \eta)}_{n_2}$ . So, from Theorem 4.1, under the given condition,  $F_{Y_{n:n}}(x)/F_{X_{n:n}}(x)$  is

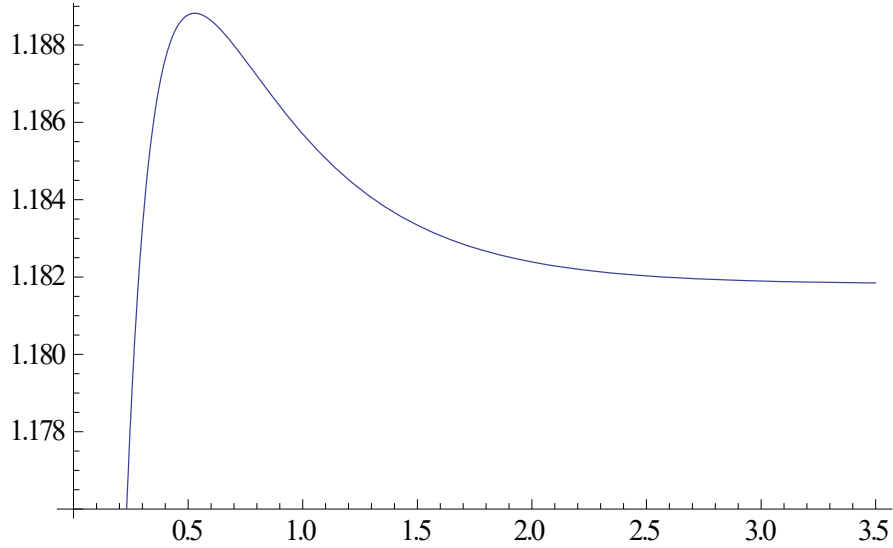


Figure 3: Curve  $\tilde{r}_{Y_{2:2}}(x)/\tilde{r}_{X_{4:4}}(x)$  for  $x > 0$ .

increasing in  $x > 0$ . Again Theorem 4.3 gives that under the given condition,  $\tilde{r}_{Y_{n:n}}(x)/\tilde{r}_{X_{n:n}}(x)$  is increasing in  $x > 0$ . Hence the theorem follows.

**Corollary 4.3** *Let  $X_1$  and  $X_2$  be independent following PO model with parameters  $\lambda_1$  and  $\eta$  respectively, and let  $Y_1$  and  $Y_2$  be independent following PO model with parameters  $\mu_1$  and  $\eta$  respectively. Then*

$$\lambda_1 \leq \eta \leq \mu_1 \Rightarrow X_{2:2} \leq_{lr} Y_{2:2}.$$

## 5 Conclusion

In this paper, we have studied stochastic comparison of series and parallel systems formed from independent heterogeneous components having lifetimes following the PO model. Most of the results are obtained using different concepts of majorization. We have also compared a system formed of heterogeneous components with another system of homogeneous components. We have derived conditions under which two series systems with heterogeneous components are ordered with respect to different stochastic orders; in case of multiple-outlier model, they are compared with respect to likelihood ratio order and relative ageing in terms of hazard rate. We have also derived conditions under which a series system with heterogeneous components and that with homogeneous components are ordered with respect to the above mentioned stochastic orderings. In case of parallel system, we have obtained conditions under which two parallel systems with heterogeneous components are ordered with respect to usual stochastic order and reversed hazard rate order. The comparison is also made in case of a parallel system

with heterogeneous components and that with homogeneous components. However, unlike series system, with suitable counterexamples we have showed that even in case of multiple-outlier model, under majorization order, two parallel systems with heterogeneous components may not be comparable with respect to likelihood ratio order and relative ageing in terms of reversed hazard rate, although, under more restricted conditions, we are able to compare the parallel systems with respect to those stochastic orderings.

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